DISTRIBUTIONS OF RATIONAL POINTS ON KUMMER VARIETIES

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ABSTRACT. We prove several results on the number of rational points on open subsets of Kummer varieties of arbitrary dimension. Some of our results are unconditional, and others depend on the Parity Conjecture (a corollary of the Conjecture of Birch and Swinnerton-Dyer). As examples, we show that (conditional on the Parity Conjecture) all odd-dimension Kummer varieties over $\mathbb{Q}$ which are quotients of absolutely simple abelian varieties have dense rational points, and we construct an infinite family of K3 surfaces with dense rational points.

CONTENTS

1. INTRODUCTION

Throughout, A will denote a principally polarised abelian variety over a number field k, and $K = A/\{\pm 1\}$ its (singular) Kummer variety. We also assume that A is simple over every quadratic extension of k. Write $\pi : A \to K$ for the quotient map (of degree 2). The polarisation on A induces a canonical height h in the sense of Néron-Tate - it is a positive definite quadratic form (in particular, we are working with logarithmic heights). We assume the height to be normalised so that it is invariant under taking algebraic field extensions. Now given a point $p \in K(k^{\mathrm{alg}})$, there exists a point $q \in A(k^{\mathrm{alg}})$ such that $\pi(q) = p$, and we set $h(p) = h(q)$; note that this is independent of the choice of q.

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Given an algebraic variety X/k with a non-degenerate height function h, and l/k a finite extension, we set

$$n_{X,l,h}(B) = \#\{p \in X(k) : h(p) \le B\}.$$

If X is a subset of A or of K, then we take the height on X to be that induced from the height on A or respectively K as defined above, and we omitt it from the notation. Similarly if $l = k$ then we omitt it from the notation.

Our main results are as follows:

1) Given an integer $n > 0$, there exists a number field f/k such that on any dense open subset $U \subset K$, there exists a constant $c_U > 0$ with

$$n_{U,f}(B) > c_U B^n. \tag{1}$$

2) Suppose that the Parity Conjecture holds, and suppose that the abelian variety A has rank n and satisfies one of the conditions given in 3.3 (for example, $k = \mathbb{Q}$ and $\dim(A)$ is odd). Then for any dense open subset U of K, there exists a constant $c_U > 0$ such that

$$n_U(B) > c_U B^{1/2}. \tag{2}$$

In particular, rational points are Zariski dense.

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1.1. **Structure of the proof.** We will begin by proving some basic results relating the rational points on an open subset of K to the points on A defined over a quadratic extension of the ground field. Result (1) will follow easily from this. Next, we will discuss the Parity Conjecture, and show that the conditions in 3.3 are sufficient to force the sign of the functional equation of A to 'jump up' in some quadratic extensions, which will imply (2). Finally, we collect some easy consequences of these results, including the construction of an infinite family of K3 surfaces with (conditionally) dense rational points.

2. The basic results

The following lemma is the basic tool for constructing rational points on K:

Lemma 2.1. *Let l/k be a quadratic extension, and let $p \in A(l)$ be such that the image of p in $A(l)/A(k)$ has infinite order. Write σ for a generator of the Galois group of l over k. Write $q \overset{def}{=} p - \sigma(p) \in A(l)$. Then*

1) q has infinite order in $A(l)/A(k)$ and maps to a k-rational point on the Kummer K;

2) $\mathrm{h}(q) \le 4\,\mathrm{h}(p)$.

Proof. Clearly $\sigma(q) = -q$ so q maps to a rational point on K. To prove Assertion (1), it remains to check that q has infinite order in $A(l)/A(k)$. Well, otherwise there exists an $m > 0$ such that $mq = \sigma(mq) = -mq$, so $2mq = 0$, and hence $\sigma(2mp) = 2mp$, contradicting our assumption that p has infinite order in $A(l)/A(k)$.

Assertion (2) follows from the parallelogram law. □

We recall

Lemma 2.2. *Let A be an abelian variety, and l/k a finite extension such that the base-change A_l is simple. Let Z be a proper Zariski-closed subset of A. Then $Z(l)$ is finite.*

Proof. This follows immediately from the Mordell-Lang Conjecture, a theorem due to Faltings [Fal94]. □

Our basic result is then

Proposition 2.3. *Let A and K be as above, and $U \subset K$ dense open. Suppose that l/k is a quadratic extension, and set $r_l = \mathrm{rk}(A(l))$, $r_k = \mathrm{rk}(A(k))$, $r = \max(r_k, r_l - r_k)$. Then there exists a constant $c > 0$ such that*

$$n_U(B) > cB^{r/2}. \tag{3}$$

Proof. It is standard that $n_A(B) \gg B^{r_k/2}$. By Lemma 2.2 we know that only finitely many points of $A(k)$ fail to map to points of U, and since the canonical map $\pi : A \to K$ has degree 2, we are done.

It remains to show that $n_U(B) \gg B^{(r_l - r_k)/2}$. Writing σ for the generator of $\mathrm{Gal}(l/k)$ we set

$$S = \{p - \sigma(p) : p \in A(l)\}, \tag{4}$$

giving an exact sequence

$$0 \to A(k) \to A(l) \to S \to 0, \tag{5}$$

so $\mathrm{rk}(S) = r_l - r_k$. Thus

$$\#S(B) \gg B^{(r_l - r_k)/2}, \tag{6}$$

and by the same arguments as before, we see that only finitely many points of S fail to map to points of U, and π has finite degree, so we are done. □

Proof of Result (1). Result (1) now follows easily from the observation that for any $r > 0$, we can find a finite extension f/k such that $A(f)$ has rank at least $2(r + \mathrm{rk}(A(k)))$; indeed, this result needs far less than we have shown above. □

3. Parity and ranks

The Conjecture of Birch and Swinnerton-Dyer predicts that the order of vanishing at $s = 1$ of the L-series associated to an abelian variety is equal to its rank. Reducing this modulo 2, we should find that the parity of the rank is equal to the sign of the function equation of the L-series. This statement has the advantage that, although the analytic continuation of the L-series to $s = 1$ is only conjectural, the sign of the functional equation (if the continuation exists) is given by the root number, whose existance is not conjectural. Thus it is possible to state the Parity Conjecture, whose statement does not depend on the existence of analytic continuations of L-functions:

Conjecture 3.1. *Let X be an abelian variety over a number field, with rank r and global root number $w(X)$ (as defined for example in* [DD09]*). Then*

$$w(X) \equiv r \mod 2.$$

We will apply this to the base change of A to quadratic extensions of k. We will control the root number by local considerations, and use this to force its parity to change as we go from k to a quadratic extension. This will force the rank to increase, which was what we wanted. We write

$$w(X) = \prod_{\nu} w(X)_\nu \tag{7}$$

where the product is over a proper set of absolute values for k, and $w(X)_\nu$ denotes the local root number at ν, as defined in [DD09].

Lemma 3.2. *Let l/k be a finite extension. Fix a place ν of m.*

1) If ν is non-Archimedean, suppose the Néron model of A has split-semistable reduction over ν. Write t for the rank of the (split) toric part of the connected component of the Néron model of $X/\mathcal{O}_k$. Then for any place ω of l dividing ν, the local root number is given by

$$w(A/l)_\omega = (-1)^t. \tag{8}$$

2) If ν is Archimedian, then

$$w(A/l)_\omega = (-1)^{\dim(A)}. \tag{9}$$

Proof. 1) Applying the formula in [DD09, Proposition 3.23] to the trivial representation yields

$$w(A/l)_\omega = (-1)^{\langle 1, X(\mathcal{T}^*)\rangle}, \tag{10}$$

where $X(\mathcal{T}^*)$ is the character group of the dual of the (split) toric part of the connected component of the identity of the Néron model of A/l. This is a direct sum of t copies of the trivial representation (since the rank is invariant under finite extensions, and torus is split), and hence the inner product evaluates to t as desired.

2) This follows immediately from [Sab05, Lemma 3.1.1, p31]. $\square$

In order to apply this result to force the global root number to change sign, we need to impose certain conditions on our abelian variety.

Condition 3.3. *The abelian variety A/k is required to satisfy at least one of the following conditions:*

1) A has at least one non-Archimedean place with split-semistable reduction and where the rank of the toric part of the Néron model is odd;

2) $\dim(A)$ is odd, and k has at least one real place.

Proposition 3.4. *Let S_1, S_2 be two finite disjoint sets of places of k. Then there exists an infinite set $T(S_1, S_2)$ of quadratic extensions $l = k(\sqrt{d_l})$ such that*

1) $\forall \nu \in S_1$ and $l \in T(S_1, S_2)$, ν splits in l;

2) $\forall \nu \in S_2$ and $l \in T(S_1, S_2)$, ν does not split in l;

Proof. Whether a prime ν splits in l depends on whether d_l is a non-zero square in the residue field of ν (for non-Archimedian ν), or whether $\nu(d_l)$ is positive (for real Archimedean ν). □

Proposition 3.5. *Assume the Parity Conjecture holds for the base-change of A to any quadratic extension of k. Then there exists quadratic extension $l = k(\sqrt{d_l})/k$ satisfying $\mathrm{rk}(A(k)) < \mathrm{rk}(A(l))$.*

Proof. By assumption, A satisfies one of the conditions in 3.3. As such, there exists a place ν_0 of k such that for all l/k, and for all places ω of l dividing ν_0, we have that $w(A/l)_\omega$ is odd. We also have a finite set S_2 of places of k such that for all places ν of k outside $S_2 \cup \{\nu_0\}$, and for all finite l/k and places ω of l dividing ν, we have that $w(A/l)_\omega$ is even.

Applying Proposition 3.4, we obtain an infinite set $\tilde{T}$ of quadratic extensions such that in each extension, ν_0 splits and no place in S_2 splits. As such, for each $l \in \tilde{T}$, we find that the *global* root number of A over l is different from the global root number of A over k. Under the parity conjecture, this forces the rank of A over each such l to be different from the rank over k; the rank cannot decrease, so it must increase, as desired. Picking any l in $\tilde{T}$, we are done. □

From this and Proposition 2.3, our second main result follows immediately.

4. Examples

In this section, we note some simple consequences of the above results, and construct some explicit examples.

4.1. **Consequences of unbounded ranks.** It is an open question whether the ranks of quadratic twists of a fixed abelian variety are

bounded. Suppose that the ranks of quadratic twists of A are unbounded. Then on any dense open subset $U \subset K$, and for any $n > 0$, there exists a constant $c_{U,n} > 0$ such that

$$n_U(B) > c_{U,n} B^n. \tag{11}$$

To deduce this, it suffices to recall that quadratic twists A_l of A are constructed by descent from the base-change of A to quadratic extensions l/k, and that in this setup we have $\mathrm{rk}(A(k)) + \mathrm{rk}(A_l(k)) = \mathrm{rk}(A(l))$. In particular, the conjecture implies that the rank of A over quadratic extensions l/k is unbounded, and so the result follows immediately from Proposition 2.3.

4.2. **Odd dimension.** An odd-dimensional Kummer variety that is the quotient of an abelian variety simple over all quadratic extensions clearly satisfies Condition (1), and so (assuming the Parity Conjecture) has rational points growing at least like $cB^{1/2}$. Absolutely simple abelian varieties are Zariski dense in the moduli space of polarised abelian varieties, and hence we may deduced from the Parity Conjecture that 'most' odd dimensional Kummers have rational points growing at least like $cB^{1/2}$.

4.3. **Dimension 2.** Desingularised Kummer surfaces are examples of K3 surfaces. As we are concerned with rational points on open subsets, we may disregard the desingularisation, and so our results have consequences for the distribution of rational points on these K3 surfaces.

Since the dimension is even, we must use Condition (2) to construct examples. The reduction type is a local condition, and we will see below that simplicity of A can also be enforced locally, allowing us to easily construct infinite families of K3 surfaces whose rational points grow at least like $cB^{1/2}$, assuming the Parity Conjecture. We give one such construction in the remainder of this note.

4.3.1. *The construction.* Let q be a prime not dividing 66. Let C be an odd-degree hyperelliptic curve of genus 2 over $\mathbb{Q}$ given by an equation $y^2 = f(x)$, with $f \in \mathbb{Z}[x]$ a monic integral polynomial with non-vanishing discriminant Δ. Suppose f is of the form

$$x^5 + ax^2 + b \tag{12}$$

where $a \in 1 + p\mathbb{Z}_p$, $b \in p + p^2\mathbb{Z}_p$ and $a, b \in 1 + 11\mathbb{Z}_{11}$ (the reasons for these conditions will become clear later on).

Let A denote the Jacobian of C, and K the corresponding singular Kummer surface. Following [FS97], K may be embedded in $\mathbb{P}^3$ with coordinates k_1, k_2, k_3, k_4 as a quartic hupersurface given by the equation

$$(k_2^2-4k_1k_3)k_4^2+(-4k_1^3b+2k_1^2k_3a+k_2k_3^2)k_4-4k_1^4ab+4k_1^2k_2k_3b-4k_1k_2^3b-4k_1k_2k_3^2a+k_3^4.$$

We remark that our height h coincides with the naive height on $\mathbb{P}^3$ up to $O(1)$, as noted in [FS97].

Claim 4.1. *The abelian variety A is simple over every quadratic extension $l/\mathbb{Q}$.*

Proof. We immitate the proof of Proposition 9 of [FPS97]; see also [Sto96]. We see that C and hence A has good reduction over $p = 11$, and hence the reduction map embedds $\operatorname{End} A_p$ into $\operatorname{End} A_{\overline{\mathbb{F}}_p}$. The characteristic polynomial of the Frobenius endomorphism π_p on A is given by

$$P = X^4 - tX^3 + sX^2 - ptX + p^2 = X^4 + 6X^3 - 108X^2 + 66X + 121, \tag{13}$$

where

$$t = p + 1 - \#C(\mathbb{F}_p) = -6, \tag{14}$$

and

$$s = \frac{1}{2}((\#C(\mathbb{F}_p))^2 - \#C(\mathbb{F}_{p^2})) + p - (p+1)\#C(\mathbb{F}_p) = -108. \tag{15}$$

It is easy to check that this characteristic polynomial is irreducible, and so A is simple and $\mathbb{Q}(\pi_p) \stackrel{\text{def}}{=} \mathbb{Q}[X]/P(X)$ is a field. To check that A remains simple in every quadratic extension of $\mathbb{Q}$, we could repeat this calculation replacing $\mathbb{F}_p$ by its unique degree-2 extension, but it is perhaps easier (and equivalent) to check that $\mathbb{Q}(\pi^2)$ does not lie in a proper subfield of $\mathbb{Q}(\pi)$, which can easily be checked using Sage, MAGMA or similar. □

Claim 4.2. *The abelian variety A has split semistable reduction over q, with odd-dimensional toric part.*

Proof. We consider the projective closure $\mathcal{C}$ of the curve C over $\mathbb{Z}_q$. The only non-smooth point is the node at the ideal (x, y, q), and $\mathcal{C}$ is regular at that point since the dimension of the tangent space to that ideal is two. The node has rational tangent directions ± 1, and hence the reduction is split semistable. To see that the rank of the toric part is odd, we note that the reduction graph is a loop, and hence by [BLR90, Chapter 9, Example 8, Page 246] that the toric part has rank 1. □

Thus we may conclude that, assuming the Parity Conjecture, all of the K3 surfaces constructed above have rational points growing faster than $cB^{1/2}$ on every dense open subset; in particular, rational points are Zariski dense.

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